Statistical mechanics of magnetohydrodynamics

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A statistical mechanical formulation for the steady state of self-organized magnetohydrodynamic plasma is studied based on the empirical variational principle, $\delta(E-\lambda H)=0$, for the steady state, where E and H denote the energy and the helicity of a magnetic field. The eigenfunctions of the curl operator are shown to span the phase space of a magnetic field in a bounded system, and the invariant measure is found. The classical ensemble theory is formulated assuming the Shannon or Rényi entropy. To avoid the divergence of the expectation values at nonzero temperature, Bose-Einstein statistics is also phenomenologically treated. It is implied that the spectra of the energy, helicity, and the helicity fluctuation obey the power law for a multiply connected domain with a nonzero cohomological field. For the toroidal system, these powers are implied to be three, three, and two, respectively. The invariant measure for the incompressible flow in a bounded domain is also given.

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I. INTRODUCTION

The magnetohydrodynamical systems generate macroscopically ordered states from random disordered states. These phenomena are primarily due to the dynamical laws of such systems, that is, the magnetohydrodynamics equations (MHD equations). These equations are, however, not simple and they determine the behavior of the system more precisely than we expect. What we want to know is not a microscopic structure which fluctuates much under the change of minute conditions but the macroscopic coarse-grained structure which is stable under the microscopic changes.

Such separation of the scale is usually impossible. Strong experimental or mathematical conditions are indispensable. In some MHD systems, its self-organization phenomena seem to allow us to postulate the possibility of self-contained and self-consistent descriptions in the macroscopic level without referring to the microscopic details. More explicitly, we have a quantitative phenomenological variational principle which determines the macroscopic structure of a magnetic field [1,2]:

$$\delta(E-\lambda H)=0, \tag{1.1}$$

where E and H denote the energy and the helicity of a magnetic field, respectively. This variational principle first appeared when Chandrasekhar and Woltjer [1] discussed the minimum energy state of magnetic flux tubes tangled in a stellar plasma with introducing the magnetic helicity to characterize the twist of magnetic fields. With a fixed gauge, we write the magnetic field $\boldsymbol{B}=\boldsymbol{\nabla}\times\boldsymbol{A}$. The helicity density is $h=\boldsymbol{A}\cdot\boldsymbol{B}$, and the helicity in a fixed domain Ω is $H=\int_\Omega h dx$. He assumed that the plasma relaxes into the minimum energy state with a given (prescribed) helicity. In a low-pressure charge-neutral plasma, the energy is dominated by $E=(2\mu_0)^{-1}\int_\Omega B^2 dx$ (μ_0, vacuum permeability). By formal calculations of the variation with appropriate boundary conditions, the minimum energy state is shown to satisfy the Beltrami condition

$$\boldsymbol{\nabla}\times\boldsymbol{B}=\lambda\boldsymbol{B}, \tag{1.2}$$

where λ is a real constant (corresponding to the Lagrange multiplier). Since the current density $\boldsymbol{j}=(\mu_0)^{-1}\boldsymbol{\nabla}\times\boldsymbol{B}$ in steady state, (1.2) implies the force-free condition ($\boldsymbol{j}\times\boldsymbol{B}=\boldsymbol{0}$), which had been considered to be obeyed by the relaxed magnetic field in a plasma [3]. If λ is an eigenvalue of the curl operator, Eq. (1.2) implies that $\boldsymbol{B}$ is the corresponding eigenfunction. And it was shown that the state becomes unstable when $|\lambda|>\lambda_{\rm min}$ [4], where $\lambda_{\rm min}$ is the nonzero and minimum absolute value of the curl eigenvalues, without charging some conditions which fix the modes with absolutely smaller eigenvalues than λ. This $\lambda_{\rm min}$ is proved to be positive (nonzero) [5]. So theoretically and experimentally interesting problems are the state for $0<|\lambda|<\lambda_{\rm min}$.

Exactly the same equation as (1.2) was found to describe the relaxed state of turbulent plasmas in laboratory experiments. Taylor [2] conjectured that a selective dissipation of the magnetic energy with respect to the helicity yields such a relaxed state. By Maxwell's equations, we obtain "Poynting's law" for the helicity,

$$\partial_t h=-\boldsymbol{\nabla}\cdot(\phi\boldsymbol{B}+\boldsymbol{E}\times\boldsymbol{A})+2\boldsymbol{E}\cdot\boldsymbol{B}, \tag{1.3}$$

where $\boldsymbol{E}$ ($=-\partial_t\boldsymbol{A}-\boldsymbol{\nabla}\phi$) is the electric field and ϕ is the scalar potential. Assuming a perfectly conductive wall at the boundary $\partial\Omega$, we obtain, using (1.3),

$$\frac{d}{dt}H=\int_\Omega 2\boldsymbol{E}\cdot\boldsymbol{B}dx. \tag{1.4}$$

In a highly conductive hydrodynamic plasma, $E_{\|} = \boldsymbol{E} \cdot \boldsymbol{B}/|\boldsymbol{B}| \approx 0$, and hence H is conserved. Furthermore, under the perfectly conductive boundary condition, $\boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{0}$, the helicity can be formulated to be a gauge invariant quantity. The conservation of the helicity imposes an essential restriction on the dynamics of the plasma. If H remains constant while the magnetic energy E achieves its minimum, the relaxed state is characterized by the minimizer of $F = E - \lambda' H$, and the formal Euler-Lagrange equation becomes (1.2).

Thê dynamical process of the relaxation was studied by computer simulations based on three-dimensional magnetohydrodynamic model equations [6].

Let us now revisit the thermodynamics and its statistical mechanics. Many experiments and speculations supported that the thermal equilibrium state is determined by the variational principle

$$\delta(F) = 0, \tag{1.5}$$

where F denotes the free energy. The thermodynamics itself is a self-consistent and self-contained theory within the macroscopic quantities like volume, pressure, and entropy. The statistical mechanics gives the relations between the microscopic dynamics and the macroscopic thermodynamics by assuming appropriate ensembles. The Boltzmann distribution is a kind of working hypothesis. It reproduces the correct results and its mathematical structure is now accepted to be natural. The additivity of the energy and the importance of the energy as the principal integral of the equation of motion imply the Boltzmann distribution with appropriate invariant measure. So most physicists already accepted that the ensemble and the Boltzmann distribution have sufficient reason to be regarded as the *reality.*

The purpose of this paper is to elucidate the ensemble description for a MHD system starting from the formal similarity between Eq. (1.1) and Eq. (1.5). It is to propose a statistical mechanics for a MHD system. There are pioneering works [7–9] towards such statistical mechanics already, which will be discussed at the end of this paper.

In this paper, we will make a statistical treatment only for the magnetic field. The velocity field is not treated explicitly in our formalism, because it does not appear in Eq. (1.1) explicitly. The variational principle (1.1) is interpreted as the zero-(helicity)-temperature form of the thermodynamic variational principle of the helicity ensemble. Appropriate space for this purpose is analyzed in the next section and a good phase space with invariant measure is given. A related topic of this phase space is given in the Appendix. In this space, the solution of Eq. (1.1) is studied in the third section. This solution is considered as the zero-temperature ground state. The fourth section proposes a simple quantal statistics after we see that the simplest classical statistics shows difficulty. Some connections to the experimental verification of this statistical mechanics are given in the fifth section.

II. PHASE SPACE

When an equilibrium or steady state exists, there are two key steps towards the statistical mechanical transcription of a variational principle. One is to find the relevant additively conserving quantity to characterize the state. In our case, the energy and the helicity of magnetic field play this role. The other is to find the invariant measure of the temporal evolution equation. It corresponds to Liouville's theorem in the classical Hamilton mechanics. In this section, it is proved that the expansion coefficient of the magnetic field, $\boldsymbol{B}$, with the complete orthogonal functions described below is a natural phase space and its volume element is temporally invariant.

Let Ω ($\subset \mathbf{R}^3$) be a bounded domain with a smooth boundary $\partial\Omega$. We denote by $\boldsymbol{n}$ the outward unit normal vector onto $\partial\Omega$. We consider a function space of real solenoidal vector fields in Ω,

$$L^2_\sigma(\Omega) = \{\boldsymbol{u} \in L^2(\Omega); \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \text{ in } \Omega, \boldsymbol{n} \cdot \boldsymbol{u} = 0 \text{ on } \partial\Omega\}, \tag{2.1}$$

which is a Hilbert space endowed with the standard L^2 innerproduct ( ). If Ω is multiply connected, we obtain the subspace of harmonic vector fields,

$$L^2_H(\Omega) = \{\boldsymbol{u} \in L^2(\Omega); \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{0} \text{ in } \Omega, \boldsymbol{n} \cdot \boldsymbol{u} = 0 \text{ on } \partial\Omega\}, \tag{2.2}$$

which represents the cohomology class, and the dimension of this $L^2_H(\Omega)$ is equal to the first Betti number ν of Ω. We write $L^2_\sigma(\Omega) = L^2_H(\Omega) \oplus L^2_\Sigma(\Omega)$, where $L^2_\Sigma(\Omega)$ is defined as the orthogonal complement of $L^2_H(\Omega)$. For these function spaces, the following lemma is proved [5].

Lemma 1. (1) When we consider eigenvalue problem

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \lambda \boldsymbol{u}, \quad \boldsymbol{u} \in L^2_\Sigma(\Omega), \tag{2.3}$$

we obtain a complete orthogonal set of eigenfunctions to span $L^2_\Sigma(\Omega)$. All eigenvalues are real, nonzero, and discrete.

(2) For every $\boldsymbol{u} \in L^2_\sigma(\Omega)$, we have an orthogonal expansion

$$\boldsymbol{u} = \sum_j (\boldsymbol{u}, \boldsymbol{\varphi}_j)\boldsymbol{\varphi}_j + \sum_{\ell=1}^{\nu} (\boldsymbol{u}, \boldsymbol{h}_\ell)\boldsymbol{h}_\ell, \tag{2.4}$$

where $\boldsymbol{\varphi}_j \in L^2_\Sigma(\Omega)$ is the eigenfunction of the curl operator and $\boldsymbol{h}_\ell$ is the orthogonal basis of $L^2_H(\Omega)$.

In the following, the subscript j for the nonzero eigenvalue and its eigenfunction of curl operator is assumed to run over all integers except zero, and this numbering is assumed to follow the order of the eigenvalue. Negative and positive subscripts are assumed to correspond to negative and positive eigenvalues, respectively. That is, the eigenvalue numbering looks like

$$\cdots \le \lambda_{-3} \le \lambda_{-2} \le \lambda_{-1} < 0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots. \tag{2.5}$$

These are unbounded and go to $\pm\infty$ when $j \to \pm\infty$.

Now we can find the phase space of the magnetic field.

Lemma 2. Let $\boldsymbol{v}(x,t)$ be a smooth vector field in Ω. Suppose that a solenoidal vector field $\boldsymbol{f}(x,t)$ obeys

$$\partial_t \boldsymbol{f} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{f}) \quad \text{in } \Omega, \tag{2.6}$$

$$\boldsymbol{n} \times (\boldsymbol{v} \times \boldsymbol{f}) = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{2.7}$$

Using the eigenfunctions of the curl operator, we expand

$$\boldsymbol{f}(x,t) = \sum_j c_j(t)\boldsymbol{\varphi}_j(x) + \sum_{\ell=1}^{\nu} \check{c}_\ell(t)\boldsymbol{h}_\ell(x) \tag{2.8}$$

(see Lemma 1). Then, $dC = d\check{c}_1 \cdots d\check{c}_\nu \prod_j dc_j$ is an invariant measure.

In fact, by the boundary condition (2.7), we observe

$$\frac{d}{dt}\check{c}_\ell = 0 \quad (\forall \ell). \tag{2.9}$$

Using (2.6) and (2.7), we obtain

$$\begin{aligned} \frac{d}{dt}c_j &= (\boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{f}), \boldsymbol{\varphi}_j) = (\boldsymbol{v} \times \boldsymbol{f}, \boldsymbol{\nabla} \times \boldsymbol{\varphi}_j) \\ &= \lambda_j (\boldsymbol{v} \times \boldsymbol{f}, \boldsymbol{\varphi}_j) \\ &= \lambda_j \left[\sum_k c_k (\boldsymbol{v} \times \boldsymbol{\varphi}_k, \boldsymbol{\varphi}_j) + \sum_{\ell=1}^{\nu} \check{c}_\ell (\boldsymbol{v} \times \boldsymbol{h}_\ell, \boldsymbol{\varphi}_j) \right]. \end{aligned} \tag{2.10}$$

Since $(\boldsymbol{v} \times \boldsymbol{\varphi}_j) \cdot \boldsymbol{\varphi}_j \equiv 0$, we find

$$\partial(dc_j/dt)/\partial c_j = 0 \quad (\forall j). \tag{2.11}$$

Hence the measure dC is invariant. This implies that these $\check{c}$ and c are a good set of coordinates in the phase space in the statistical mechanical sense.

In a MHD system, $\boldsymbol{v}$ and $\boldsymbol{f}$ are the velocity field of the fluid motion and the magnetic field, $\boldsymbol{B}$, respectively. The velocity field is now treated to be a separated freedom from the magnetic field. This treatment will be good when the magnetic field has the most energy in the system, and then the velocity field acts as a perturbation or as a fluctuation generator to the magnetic field. When we expand a magnetic field as

$$\boldsymbol{B}(x) = \sum_j c_j \boldsymbol{\varphi}_j(x) + \sum_{\ell=1}^{\nu} \check{c}_\ell \boldsymbol{h}_\ell(x), \tag{2.12}$$

the second summation term over the harmonic field in the RHS is the same for all possible $\boldsymbol{B}$ because of the boundary condition $\boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{0}$. It is called the cohomology field. So we do not treat $\check{c}_\ell$ as a dynamical variable, instead, as a constant. Only c_j's are treated as dynamical variables, and the first summation in the RHS of Eq. (2.12) is denoted by $\boldsymbol{B}_\Sigma$. The energy of this $\boldsymbol{B}$ is expressed as

$$E = \sum_j c_j^2 + \sum_{\ell=1}^{\nu} \check{c}_\ell^2. \tag{2.13}$$

The second summation, the energy of the cohomology field gives only a constant contribution. Taking $\boldsymbol{g}_\ell$ to be $\boldsymbol{h}_\ell = \boldsymbol{\nabla} \times \boldsymbol{g}_l$, the vector potential of $\boldsymbol{B}$ is

$$\boldsymbol{A}(x) = \sum_j \frac{c_j}{\lambda_j}\boldsymbol{\varphi}_j(x) + \sum_{l=1}^{\nu} \check{c}_l \boldsymbol{g}_l(x). \tag{2.14}$$

We can add any function in $L^2_H(\Omega)$ to vector potentials of the cohmology field, $\sum_l \check{c}_l \boldsymbol{g}_l(x)$. This corresponds to the gauge degree of freedom.

The relative helicity is defined by

$$\int_\Omega \boldsymbol{A} \cdot \boldsymbol{B}_\Sigma dx = \sum_j \left(\frac{c_j^2}{\lambda_j} + L_j c_j \right), \tag{2.15}$$

where

$$L_j = \sum_{\ell=1}^{\nu} \check{c}_\ell \Delta_{j,\ell} \tag{2.16}$$

and

$$\Delta_{j,\ell} = (\boldsymbol{\varphi}_j, \boldsymbol{g}_\ell). \tag{2.17}$$

The $\Delta_{j,\ell}$ may be called the cohomology-helicity coupling constant. The L_j is named the cohomology coefficient. The difference between the relative helicity and the helicity is a constant determined only by the cohomology field and its vector potential, so we can neglect it. It should be remarked that the relative helicity (2.15) is gauge invariant because of the perfectly conductive boundary condition. In the following, we will only use this gauge-invariant quantity for the helicity and call it simply the helicity.

III. SOLUTION OF THE VARIATIONAL PROBLEM

Now we can solve the variational problem (1.1). By using the expansion Eq. (2.12), this problem becomes

$$\begin{aligned} 0 &= \delta_{\{c_j\}} \sum_j \left[\left(1 - \frac{\lambda}{\lambda_j}\right) c_j^2 - \lambda L_j c_j \right] \\ &= \delta_{\{c_j\}} \sum_j \left[\left(1 - \frac{\lambda}{\lambda_j}\right) \left(c_j - \frac{\lambda\lambda_j L_j}{2(\lambda_j - \lambda)}\right)^2 \right. \\ &\quad \left. - \frac{\lambda^2 \lambda_j L_j^2}{4(\lambda_j - \lambda)} \right]. \end{aligned} \tag{3.1}$$

For $0 < \lambda < \min_j |\lambda_j|$, the solution is

$$c_j^0 = \frac{\lambda\lambda_j L_j}{2(\lambda_j - \lambda)} \quad (\forall j). \tag{3.2}$$

The L_j will decay algebraically in terms of j for large $|j|$. The eigenvalue λ_j will be distributed uniformly for large $|j|$. And we can expect that the summation $\sum_j c_j^0 \boldsymbol{\varphi}_j$ converges uniformly and absolutely. But we cannot always expect such convergence for the termwisely differentiated series $\sum_j c_j^0 \lambda_j \boldsymbol{\varphi}_j$. The energy and the helicity are expressed as

$$E = \sum_j \frac{\lambda^2 \lambda_j^2 L_j^2}{4(\lambda_j - \lambda)^2} \tag{3.3}$$

and

$$H = \sum_j \frac{\lambda\lambda_j L_j^2}{4} \frac{2\lambda_j - \lambda}{(\lambda_j - \lambda)^2}. \tag{3.4}$$

L_j is expected to show algebraic decay in terms of $1/|j|$. For example, the decay speed is expected to be the order of $1/|j|^{3/2}$ when Ω is inside of a torus (T^2). For large $|j|$, j can be regarded as the wave number k in the Fourier analysis case except a constant with the dimensionality of $(\text{length})^{-1}$, because the local feature of the sufficiently high mode will not be sensitive to the boundary condition.

IV. ENSEMBLE

MHD fluid relaxes to a kind of steady state after it starts to develop from a given initial condition. During this relaxation, the change of the magnetic helicity is slow, and it can be neglected. The energy of the magnetic field, however, dissipates largely in the early state and finally its change is also negligible in the steady state [2]. The variational principle (1.1) determines the structure of such a steady state [we should say that the validity of the expression "steady state" comes out of the success of Eq. (1.1)] and our purpose is to propose a microscopic model, which we call a "statistical mechanics of MHD," to reproduce this principle. In our terminology, the thermodynamics of MHD, Eq. (1.1), suggests that the energy E and helicity H are the relevant state variables. H is easily controlled by external condition, but E is not as we described above. So the parameter $1/\lambda$ works like a chemical potential of the grand canonical ensemble. The limitation of this chemical potential interpretation is that these E and H are defined in the same phase space.

These E and H are additive quantities in the relaxed state. So the possible distribution consistent with Eq. (1.1) is determined by specifying the information measure. When we use the Shannon entropy, $S(p) = -\sum p \ln p$, the Boltzmann distribution form in terms of these quantities appears:

$$P(E,H) \propto \exp(-\alpha H - \beta E), \tag{4.1}$$

where α and β are constants, and these E and H are microscopically defined dynamical quantities, not macroscopic. This expression is equivalent to

$$P(E,H) \propto \exp[-\beta(E - \lambda H)]. \tag{4.2}$$

This λ is adjusted to the notation in Eq. (1.1). The β is interpreted as an inverse temperature of the magnetic field, and Eq. (1.1) corresponds to the case of large β.

A more general information measure is the Rényi entropy [10],

$$S_q(\{p_i\}) = \frac{1}{1-q} \ln\left(\sum_i p_i^q\right), \tag{4.3}$$

where q is a positive parameter. The Shannon entropy is included in the Rényi entropy in the limit of $q \to 1$. The canonical distribution based on this entropy, that is, the Tsallis distribution [12], also produces a similar result to the Boltzmann distribution as is shown below.

Although the helicity H is introduced in the Boltzmann distribution function, this naive classical statistical mechanics causes the same kind of catastrophe as what we meet in the classical treatment for the blackbody radiation.

A. Classical statistics

In the preceding section, it is proved that the volume element $\prod_j dc_j$ is temporally conserved when the flow velocity field is prescribed. The proof of this, moreover, shows that each dc_j is conserved. So we concentrate on a single mode, denoted by j, first. The helicity and the energy of this mode are $c_j^2/\lambda_j + L_j c_j$ and c_j^2, respectively. The Boltzmann distribution for this amplitude c_j is

$$P_j(c) \propto \exp\left[-\beta\left(c_j^2 - \frac{\lambda}{\lambda_j}c_j^2 - \lambda L_j c_j\right)\right]. \tag{4.4}$$

In the variational principle, Eq. (1.1), we can assume the condition $0 < \lambda < \min_j |\lambda_j|$. Assuming that β is positive, the distribution function is

$$P_j(c)dc = \sqrt{\frac{\lambda_j}{\pi\beta(\lambda_j - \lambda)}} \times \exp\left[-\beta\left(1 - \frac{\lambda}{\lambda_j}\right)(c_j - c_j^0)^2\right] dc, \tag{4.5}$$

where c_j^0 is defined in Eq. (3.2) as the solution of Eq. (1.1). In the following, the ensemble averaged value is denoted by $\langle\,\rangle$. The expectation value of the energy for this mode is

$$\langle c_j^2 \rangle = \frac{\lambda_j}{2\beta(\lambda_j - \lambda)} + (c_j^0)^2 = \frac{\lambda_j}{2\beta(\lambda_j - \lambda)} + \frac{\lambda^2\lambda_j^2 L_j^2}{4(\lambda_j - \lambda)^2}. \tag{4.6}$$

The helicity is

$$\left\langle \frac{c_j^2}{\lambda_j} + L_j c_j \right\rangle = \frac{1}{2\beta(\lambda_j - \lambda)} + \frac{(c_j^0)^2}{\lambda_j} + L_j c_j^0 = \frac{1}{2\beta(\lambda_j - \lambda)} + \frac{\lambda\lambda_j L_j^2}{4}\frac{2\lambda_j - \lambda}{(\lambda_j - \lambda)^2}. \tag{4.7}$$

The distribution over our phase space, $\{c_j\}$, is simply the product for each P_j. So the average of E and H should be also simply obtained by summing up over all modes:

$$\langle E \rangle = \sum_j \left[\frac{\lambda_j}{2\beta(\lambda_j - \lambda)} + \frac{\lambda^2\lambda_j^2 L_j^2}{4(\lambda_j - \lambda)^2}\right] \tag{4.8}$$

and

$$\langle H \rangle = \sum_j \left[\frac{1}{2\beta(\lambda_j - \lambda)} + \frac{\lambda\lambda_j L_j^2}{4}\frac{2\lambda_j - \lambda}{(\lambda_j - \lambda)^2}\right]. \tag{4.9}$$

In these summations, the summations of the "cohomological terms,"

$$\sum_j \frac{\lambda^2\lambda_j^2 L_j^2}{4(\lambda_j-\lambda)^2} \quad \text{and} \quad \sum_j \frac{\lambda\lambda_j L_j^2}{4}\frac{2\lambda_j-\lambda}{(\lambda_j-\lambda)^2} \tag{4.10}$$

are expected to converge, because $|L_j|$ decays faster than $1/|j|$. The summation of the first terms in the helicity may be interpreted to converge by taking the limit in the form of

$$\lim_{J\to\infty}\sum_{j=-J}^{J}\frac{1}{2\beta(\lambda_j-\lambda)}. \tag{4.11}$$

The first term in the energy, however, diverges. This term expresses the equipartition of the energy for every mode.

The Tsallis distribution for c_j is

$$P_j(c) \propto \left[1-\beta(q-1)\left(1-\frac{\lambda}{\lambda_j}\right)(c-c_j^0)^2\right]^{1/(q-1)}, \tag{4.12}$$

and the range of c is limited to $c_j^0 - c_j^{\max} \le c_j \le c_j^0 + c_j^{\max}$, where $1/c_j^{\max} = \sqrt{\beta(1-\lambda/\lambda_j)}$. Including the normalization factor, it becomes

$$P_j(c)dc = \frac{1}{c_j^{\max}B(1/2,q/(q-1))}\left[1-\beta(q-1)\left(1-\frac{\lambda}{\lambda_j}\right)\times(c-c_j^0)^2\right]^{1/(q-1)} dc, \tag{4.13}$$

where $B(\,)$ denotes the beta function. So the energy for each mode is calculated to be

$$\langle c_j^2\rangle = \frac{\lambda_j}{(3q-1)\beta(\lambda_j-\lambda)} + (c_j^0)^2 \tag{4.14}$$

and the helicity to be

$$\left\langle \frac{c_j^2}{\lambda_j} + L_j c_j \right\rangle = \frac{1}{(3q-1)\beta(\lambda_j-\lambda)} + \frac{(c_j^0)^2}{\lambda_j} + L_j c_j^0. \tag{4.15}$$

So the difference between the Boltzmann and Tsallis distributions reduces to a factor in front of β. Therefore the choice of the entropy is irrelevant.

B. Bose-Einstein statistics

The statistical mechanics of the blackbody radiation suggests that the quantization of the field is necessary to avoid the divergence of energy we met above. But it has not succeeded yet in our MHD equation case. This difficulty can be observed in Eq. (2.6). This evolution equation is linear in field $\boldsymbol{f}$ but the velocity field of the plasma flow, $\boldsymbol{v}$, will also evolve with the same time scale. And its evolution obeys a complicated nonlinear equation, for example, the Euler equation (A1) even within the incompressible approximation.

In spite of such difficulty for the legitimate approach to the second quantization, the magnetic field of a steady state in self-organized plasma is determined by the variational principle, (1.1), in which the flow does not appear. And the purpose of the present study is to make up a statistical mechanical formulation which reproduces this variational principle in a limit. Following is one of the simplest formulations to avoid the Rayleigh-Jeans-like catastrophe

The exponentiated factor, $(1-\lambda/\lambda_j)c_j^2 - \lambda L_j c_j$, is regarded as a transformed expression of a kind of effective Hamiltonian by replacing the canonical momentum with the canonical coordinate, c_j, using an unknown effective temporal evolution equation. We do not know which part of the c_j^2 comes from the momentum, or additional momentum contribution may be hidden. We introduce an angular frequency ω_j of this jth mode. New variables, d_j, are introduced to shift the average to zero and to be normalized, that is,

$$d_j = \frac{1}{\sqrt{\omega_j}}\left[c_j - \frac{1}{2}\frac{\lambda\lambda_j L_j}{\lambda_j-\lambda}\right]. \tag{4.16}$$

The factor $1/\sqrt{\omega_j}$ is a naive normalization factor which provides the unit of the field quantum. Then we assume that this d_j is the bosonic annihilator by charging the commutation relation, $[d_i,d_j^\dagger] = \delta_{ij}$ or $[c_i,c_j^\dagger] = \omega_j\delta_{ij}$.

The Bose-Einstein statistics gives the averaged number of these quanta in the jth mode as

$$\langle n_j\rangle = \langle d_j^\dagger d_j\rangle = \frac{1}{\exp[\beta(1-\lambda/\lambda_j)\omega_j]-1}. \tag{4.17}$$

The chemical potential is taken to be zero because we quantized the magnetic field.

The expectation values of the energy and the helicity are

$$\langle c_j^\dagger c_j\rangle = \omega_j\langle n_j\rangle + (c_j^0)^2 \quad (n_j = d_j^\dagger d_j) \tag{4.18}$$

and

$$\left\langle \frac{c_j^\dagger c_j}{\lambda_j} + \frac{1}{2}L_j(c_j^\dagger + c_j)\right\rangle = \frac{\omega_j}{\lambda_j}\langle n_j\rangle + \frac{(c_j^0)^2}{\lambda_j} + L_j c_j^0. \tag{4.19}$$

The total energy and helicity are

$$\langle E\rangle = \sum_j \left[\frac{\omega_j}{\exp[\beta(1-\lambda/\lambda_j)\omega_j]-1} + \frac{\lambda^2\lambda_j^2 L_j^2}{4(\lambda_j-\lambda)^2}\right] \tag{4.20}$$

and

$$\langle H\rangle = \sum_j \left[\frac{\omega_j/\lambda_j}{\exp[\beta(1-\lambda/\lambda_j)\omega_j]-1} + \frac{\lambda\lambda_j L_j^2}{4}\frac{2\lambda_j-\lambda}{(\lambda_j-\lambda)^2}\right]. \tag{4.21}$$

The ω_j will diverge when $|\lambda_j|$ diverges, and $|L_j|$ will decay faster than $1/|j|$. So these expressions now converge.

V. SOME IMPLICATIONS

The Bose-Einstein-type statistical mechanical formalism proposed in the preceding section is a theoretically naive and simple extension of the variational principle (1.1) so that fluctuations around the variationally determined state can be described. Experimental verification is expected and some characteristic predictions of the present formalism are shown in this section for that purpose.

The fluctuations of energy and helicity are

$$\langle(\Delta E)^2\rangle = \sum_j \omega_j^2 \langle(\Delta d_j^\dagger d_j)^2\rangle = \sum_j \omega_j^2 \frac{\exp[\beta(1-\lambda/\lambda_j)\omega_j]}{\{\exp[\beta(1-\lambda/\lambda_j)\omega_j]-1\}^2} \quad (5.1)$$

and

$$\langle(\Delta H)^2\rangle = \sum_j \left[\frac{\omega_j^2}{\lambda_j^2}\langle(\Delta d_j^\dagger d_j)^2\rangle + \frac{L_j^2\omega_j}{4}\left(\frac{\lambda_j}{\lambda_j-\lambda}\right)^2 (2\langle d_j^\dagger d_j\rangle + 1)\right] = \sum_j \left[\frac{\omega_j^2}{\lambda_j^2}\frac{\exp[\beta(1-\lambda/\lambda_j)\omega_j]}{\{\exp[\beta(1-\lambda/\lambda_j)\omega_j]-1\}^2} + \frac{L_j^2\omega_j}{4}\left(\frac{\lambda_j}{\lambda_j-\lambda}\right)^2 \frac{\exp[\beta(1-\lambda/\lambda_j)\omega_j]+1}{\exp[\beta(1-\lambda/\lambda_j)\omega_j]-1}\right], \quad (5.2)$$

respectively. The summation of this second term will converge. For a torus, for example, if ω_j does not grow faster than j^2, it converges. The energy fluctuation decays exponentially for higher modes. But the power-law spectrum is predicted for the helicity fluctuation from its second term in Eq. (5.2). The power exponent is determined from j dependence of $L_j^2\omega_j$ for large $|j|$.

Spatial correlation of the magnetic field can be derived. For example, two point correlation is expressed as

$$\langle \boldsymbol{B}(x)\boldsymbol{B}(y)\rangle = \sum_j [\omega_j\langle d_j^\dagger d_j\rangle + (c_j^0)^2](\boldsymbol{\varphi}(x), \boldsymbol{\varphi}(y)). \quad (5.3)$$

It is straightforward (but tedious) to get more explicit expression of this kind of spatial correlations.

The power-law behavior of the helicity fluctuation with its exponent is simple and characteristic in the present quantal statistical mechanics.

VI. SUMMARY AND DISCUSSION

A statistical mechanical formulation for the self-organized MHD fluid is proposed. It is a naive extension of the variational principle (1.1) which suggests that the structure of the magnetic field is relevant to the steady state structure of such MHD fluid. So the velocity field is neglected in the present formalism.

It is shown that the eigenfunctions of curl span a convenient phase space when the system is bounded. For a given velocity field, the volume of the expansion coefficients is temporally invariant and this corresponds to Liouville's theorem in the classical Hamilton mechanics. For a cylindrical system, this has been already shown by Turner [9] and our present proof applies in a very general situation. Our domain covers not only a simply connected domain, but also a multiply connected domain. Furthermore, the same functional analytic space turns out to be a good phase space with invariant measure even for the incompressible flow (see the Appendix) and this fact may grow to one step of the statistical mechanical theory for the turbulent flow. Previous attempts mostly used plane wave to make phase spaces [7], and met some difficulties to reproduce the power-law spectrum.

In this phase space, the energy and helicity are used as additive conserving quantities to translate the variational principle (1.1) to the ensemble and introduce fluctuations. But the simplest classical statistics leads to the divergence of the expectation values. So some more assumptions are necessary to make a finite theory. One is to restrict the relevant modes to finite as was proposed in previous formalism [9], but the solution of the original variational principle (1.1) itself requires an infinite number of modes to reproduce its solution with the eigenfunctions of curl operator, as we have seen in the third section.

Our present formalism uses the quantal statistics by charging second quantization. The relevant functional $E-\lambda H$ including a chemical-potential-like parameter λ is interpreted as a transcription of an effective Hamiltonian of the system. The frequency of each mode is introduced artificially. The fluctuation currently relevant is, however, not large, that is, the temperature $1/\beta$ is small. So a linear dispersion approximation, $\omega_j = \gamma\lambda_j$, will be good. The implication of the present formulation for general geometry is stressed here: the ground-state structure has power-law spectrum in energy and helicity and it is also the case for the helicity "thermal" fluctuation in our statistical mechanical sense. This power-law behavior stems in the tangling of the dynamical magnetic field with the cohomological magnetic field. Now for the torus or cylinder, the power exponents are predicted to be three for the ground-state energy and helicity, and using the above linear approximation, the exponent for the thermal fluctuation of the helicity is two. The experimental observation of these power exponents will be a good test of the present formulation.

At nonzero temperature in the present sense, the variational principle will be modified to that for the thermodynamic free energy as

$$\delta(E-\lambda H - TS) = 0, \quad (6.1)$$

where T is $1/\beta$ in the present formalism. S denotes the entropy which may not have been observed because the temperature T has been small. But the measurement for the helicity fluctuation will reveal its statistical nature.

Purely theoretically, even if we accept the existence of the statistical mechanics for our problem, a very different first step is possible. For example, in the present paper, we assume only the Shannon entropy to select the distribution. But the steady state of the MHD system may reject to measure our knowledge to its subsystem. In such a case, we have to use Rényi entropy and a different distribution function [12] from the current Boltzmann type. Before going into the complicated forests, we now propose a familiar extension in this paper. The experimental verification is now expected.

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APPENDIX: INVARIANT MEASURE OF INCOMPRESSIBLE FLOW

Using the eigenfunction expansion associated with the curl operator, we also obtain an invariant measure of incompressible ideal flow. Let $\boldsymbol{u}$ be a three-dimensional flow in a bounded domain Ω, which satisfies

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \boldsymbol{F} - \boldsymbol{\nabla} p, \qquad \text{(A1)}$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0,$$

where p is the pressure, and $\boldsymbol{F}$ is a force (for example $\boldsymbol{F} = \boldsymbol{j}\times\boldsymbol{B}$). We assume that $\boldsymbol{F}$ is not an explicit function of $\boldsymbol{u}$. The mass density is normalized to 1. The boundary condition is $\boldsymbol{n}\cdot\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$. Using $(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = (\boldsymbol{\nabla}\times\boldsymbol{u})\times\boldsymbol{u} + \boldsymbol{\nabla}(u^2/2)$, we may write (A1) as

$$\partial_t \boldsymbol{u} = -(\boldsymbol{\nabla}\times\boldsymbol{u})\times\boldsymbol{u} + \boldsymbol{F} - \boldsymbol{\nabla}\check{p}, \qquad \text{(A2)}$$

where $\check{p} = p + (u^2/2)$. Let us expand

$$\boldsymbol{u} = \sum_j v_j \boldsymbol{\varphi}_j + \sum_{\ell=1}^{\nu} \hat{v}_\ell \boldsymbol{h}_\ell, \qquad \text{(A3)}$$

cf. Lemma 1. We easily verify $(\boldsymbol{\nabla}\check{p}, \boldsymbol{\varphi}_j) \equiv 0$ $(\forall j)$ and $(\boldsymbol{\nabla}\check{p}, \boldsymbol{h}_\ell) \equiv 0$ $(\forall \ell)$. We denote $F_j = (\boldsymbol{F}, \boldsymbol{\varphi}_j)$. By (A2) and $\boldsymbol{\nabla}\times\boldsymbol{\varphi}_j = \lambda_j \boldsymbol{\varphi}_j$, we observe

$$\begin{aligned}\frac{d}{dt}v_j &= -\left(\left(\sum_m \lambda_m v_m \boldsymbol{\varphi}_m\right)\times\left(\sum_n v_n \boldsymbol{\varphi}_n\right), \boldsymbol{\varphi}_j\right) + F_j \\ &= -\sum_m\sum_n \lambda_m v_m v_n (\boldsymbol{\varphi}_m \times \boldsymbol{\varphi}_n, \boldsymbol{\varphi}_j) + F_j \\ &= -\sum_{m\neq j}\sum_{n\neq j} \lambda_m v_m v_n (\boldsymbol{\varphi}_m \times \boldsymbol{\varphi}_n, \boldsymbol{\varphi}_j) + F_j. \end{aligned} \qquad \text{(A4)}$$

We thus have $\partial(dv_j/dt)/\partial v_j = 0$ $(\forall j)$. Similarly $d\check{v}_\ell$ is invariant.

The complete set of ideal incompressible MHD equations consists of (A1) and (2.6) with $\boldsymbol{f} = \boldsymbol{B}$, $\boldsymbol{v} = \boldsymbol{u}$, and $\boldsymbol{F} = \mu_0^{-1}(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}$. One thus obtains a higher dimensional invariant measure such as $\prod_j dc_j \prod_j dv_j$. In the present theory, however, we do not invoke the statistical distribution with respect to $\prod_j dv_j$. This is due to the semiempirical assertion that a finite (but small) resistivity and viscosity violate the invariance of $\prod_j dv_j$ largely, while $\prod_j dc_j$ remains almost invariant. This fact is relevant to the hypothesis of the selective conservation of the helicity in the MHD turbulence [11].

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